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SPECIAL BASES FOR S_N AND GL(n)

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ABSTRACT

The special basis in spaces of finite dimensional representation of S_N and GL(n) is constructed and its properties are studied.

Introduction

The reflection representation of the Weyl group W of a semisimple algebraic group G can be given a special basis formed by the simple root vectors with respect to a Borel subgroup, and in this basis, the matrix expressing the action of a simple reflection in W, has a particularly nice form.

In general, one can ask whether there exist such special bases for arbitrary irreducible representations of W. This problem has been discussed in [2], [3].

In the case $W = S_N$, the symmetric group on N letters, so that $G = \mathrm{SL}(N)$, a construction of such a basis is provided by a construction of T. A. Springer [6], [7].

In the first part of this paper we describe some intrinsic properties of the projective basis of Springer, that is to say the set of lines spanned by the vectors in the basis of Springer. It can be proved that these properties characterise it for $N \leqq 6$. We conjecture that they characterise a Springer projective basis for arbitrary N.

The second part of the paper poses analogous problems for the rational irreducible representations of the general linear group GL(n) over a field of characteristic 0.

Our basic observation is that for both S_N and GL(n), the set of rational irreducible representations has the structure of a partially ordered set, and by taking the product order we can introduce a partial order on the set of rational irreducible representations of the groups $S_{t_1} \times \cdots \times S_{t_r}$, $\mathrm{GL}(s_1) \times \cdots \times \mathrm{GL}(s_r)$, respectively. Suppose now G is a linearly reductive algebraic group, and that the

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set $\hat{G}$ of rational irreducible representations of G has a structure of a partially ordered set. Let W be a G-module and for each $\sigma \in \hat{G}$ let $W_\sigma \subset W$ denote the isotypic component of type σ. Put $W^\sigma = \bigoplus_{\sigma' \leq \sigma} W_{\sigma'}$, $W_0^\sigma = \bigoplus_{\sigma' < \sigma} W_{\sigma'}$, so that $W^\sigma / W_0^\sigma \simeq W_\sigma$. The family $\{W^\sigma\}_{\sigma \in \hat{G}}$ of subspaces of W defines a filtration of W by G-stable subspaces indexed by $\hat{G}$. A basis $\{w_1, \cdots, w_m\}$ of W is said to be G-compatible if, (a) for each $\sigma \in \hat{G}$, W^σ has a basis given by the vectors in $\{w_1, \cdots, w_m\}$ which it contains, and (b) for each $\sigma \in \hat{G}$, given $w_i \in (W^\sigma - W_0^\sigma) \cap \{w_1, \cdots, w_m\}$ the residue class $\bar{w}_i$ of w_i modulo W_0^σ, $\bar{w}_i$ generates an irreducible G-module having as a basis the images modulo W_0^σ of some vectors in $(W^\sigma - W_0^\sigma) \cap \{w_1, \cdots, w_m\}$.

Our results for S_N are then as follows. Let $t = (t_1, \cdots, t_e)$, $\Sigma_{i=1}^{e} t_i = N$ be any non-necessarily ordered partition of N and consider the subgroup $S^t = S_{t_1} \times \cdots \times S_{t_e} \subset S_N$. Let U be any irreducible S_N module. If we consider U as an S^t module by restricting the action of S_N to S^t then the Springer basis is S^t compatible. Similarly in the case of $\mathrm{GL}(n)$ we define a basis for every rational irreducible $\mathrm{GL}(n)$ module U which has the following property: let $s = (s_1, \cdots, s_r)$, $\Sigma_{i=1}^{r} s_i = n$, be any non-necessarily ordered partition of n, take the subgroup $G^s = \mathrm{GL}(s_1) \times \cdots \times \mathrm{GL}(s_r) \subset \mathrm{GL}(n)$ and consider U as a G^s module by restricting the action of $\mathrm{GL}(n)$ to G^s. Then the given basis for U is G^s compatible.

The paper is divided as follows. Section 1 contains a review of the Springer construction as revised in [3]. Section 2 contains the results on the symmetric group, Section 3 the results on $\mathrm{GL}(n)$. Finally, the appendix contains a result on tensor spaces which is needed in Section 3.

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§1. Review of Springer construction

In this section we shall recall some results from [3], [6], [7]. Let X, Y be algebraic varieties over $\mathbf{C}$. Let G be a reductive complex algebraic group and let $\mathcal{B}$ be the variety of Borel subgroups of G and $\pi : Y \to X$ be a $\mathcal{B}$-bundle. A fibre preserving morphism $\phi : Y \to Y$ is said to be a unipotent transformation of π if, for each $x \in X$, the restriction $\phi_x : \pi^{-1}(x) \to \pi^{-1}(x)$ of ϕ to $\pi^{-1}(x)$ is the map induced by a unipotent element in G. We denote by Y^ϕ the fixpoint set of ϕ.

Let us restrict to the case in which $G = \mathrm{SL}(2)$ so that $\mathcal{B} = P^1$, the complex projective line. Let ρ be a Riemannian metric on Y such that the restriction of ρ on any fibre $\pi^{-1}(x) \simeq S^2$ is the standard metric on S^2. Let $\tau : Y \to Y$ be the fibre preserving map such that for any $x \in X$, $\tau\,|_{\pi^{-1}(x)}$ is the antipodal involution. Let $X_0 = \{x \in X \mid \phi_x = \mathrm{Id}\}$, $Y_0 = \pi^{-1}(X_0)$. It is shown in [3] how to construct a map

$\tilde{s}: Y^\phi \to Y^\phi$ defined up to homotopy which is an involution of Y^ϕ in the homotopy category (i.e., it is such that $\tilde{s}^2$ is homotopically equivalent to the identity) and such that $s\,|_{Y_0}$ is homotopically equivalent to $\tau\,|_{Y_0}$. In particular, $\tilde{s}$ induces an involution on $H_*(Y^\phi)$, when by $H_*(\ \ )$ we denote rational homology with infinite support (see [1] for definitions), which we denote by s.

Let $\dim_C Y^\phi = d$ and let M be an irreducible component of Y^ϕ. M is said to be vertical with respect to π if, given a generic point $x \in \pi(M)$, $\pi^{-1}(x) \cap Y^\phi = \pi^{-1}(x)$, M is said to be horizontal with respect to π if, given a generic point $x \in \pi(M)$, $\pi^{-1}(x) \cap Y^\phi$ consists of a single point.

Now a basis for $H_{2d}(Y^\phi)$ is given by the homology classes of the irreducible components of Y^ϕ of dimension d.

It is easy to see that if M is such a component and we denote by $[M]$ its homology class, we have:

$$s[M] = -[M] \qquad \text{if } M \text{ is vertical,}$$

$$s[M] = [M] + \sum n(M, M')[M'] \quad \text{if } M \text{ is horizontal,}$$

M' running over the set of irreducible components of Y^ϕ other than M. Further, $n(M, M') \geqq 0$ and $n(M, M') > 0$ if and only if $K = M \cap M'$ has codimension one in M and $\pi(M') = \pi(K)$.

We recall a lemma which will be useful in Section 2, whose proof follows easily using the methods of [3].

Lemma 1.1. *Let Y, X be algebraic varieties, X irreducible. Let $\pi: Y \to X$ be a $\mathbf{P}^1$-bundle, ϕ a unipotent transformation of π, $Y^\phi \subset Y$ its fixpoint set, M_1, M_2 two irreducible components of Y^ϕ. Let s be the involution of $H_*(Y^\phi)$ defined above. Suppose M_1 is horizontal and*

$$s[M_1] = [M_1] + n[M_2] + \textit{other terms.}$$

Let $\Psi: X \to Z$ be a morphism of algebraic varieties such that its generic fibre is irreducible and $\Psi \circ \pi(M_i)$ is dense in Z, $i = 1, 2$. Let $z \in Z$ be a generic point of Z, $X_z = \Psi^{-1}(z)$, $Y_z = \pi^{-1}(X_z)$, $Y_z^\phi = Y^\phi \cap Y_z$, $M_{i,z} = M_i \cap Y_z$, $i = 1, 2$. ($M_{2,z}$ could be reducible.) Then

(a) *Y_z is a $\mathbf{P}^1$-bundle*

$$\begin{array}{c} Y_z \\ \pi_z \big\downarrow \\ X_z \end{array}$$

and ϕ defines a unipotent transformation ϕ_z of π_z such that $(Y_z)^{\phi_z} = Y_z^\phi$.

(b) *$M_{1,z}$ is an irreducible component of Y_z^ϕ and $M_{2,z}$ is a union of irreducible components. Also if s_z is the involution of $H_*(Y_z^\phi)$*

$$s_z([M_{1,z}]) = [M_{1,z}] + n[M_{2,z}] + \textit{other terms.}$$

Let us return to the general case in which $\pi : Y \to X$ is a $\mathcal{B}$-bundle, $\mathcal{B}$ being the variety of Borel subgroups in an arbitrary reductive complex algebraic group G and φ is a unipotent transformation of π. Let $B \subset G$ be a fixed Borel subgroup. For any simple root α relative to B, let $P_\alpha \supset B$ be the parabolic subgroup generated by B and $U_{-\alpha}$. Define $\mathcal{P}_\alpha$ to be the variety of parabolic subgroups conjugate to P_α and let $\bar{\pi}_\alpha : \mathcal{B} \to \mathcal{P}_\alpha$ be the canonical projection. We define an equivalence relation $\sim$ on Y by putting $y \sim y'$ if $\pi(y) = \pi(y')$ and $\bar{\pi}_\alpha(y) = \bar{\pi}_\alpha(y')$. So if we put $Y_\alpha = Y/\sim$ we get a commutative diagram

$$\begin{array}{ccc} & & Y \\ & \overset{\pi^\alpha}{\swarrow} & \downarrow \pi \\ Y_\alpha & \underset{\pi_\alpha}{\searrow} & X \end{array}$$

where π_α is a $\mathbf{P}^1$-bundle and π^α a $\mathcal{P}_\alpha$-bundle. Further, ϕ induces a fiber preserving map $\phi' : Y_\alpha \to Y_\alpha$. Let $Y_\alpha^{\phi'}$ be the fixpoint set of ϕ'. Then if we denote by $\pi'_\alpha : Y' \to Y_\alpha^{\phi'}$ the restriction of π_α to $Y_\alpha^{\phi'}$, ϕ induces a unipotent transformation ϕ_α of π'_α and $Y'^{\phi_\alpha} = Y^\phi$. Thus from the above we get homotopy involutions s_α of Y^ϕ. It can be shown that the s_α's in homology satisfy Coxeter relations so that we get an action of the Weyl group W of G on $H_*(Y^\phi)$.

In what follows we shall be mostly interested in the following three cases of the above construction, which are fully treated in [3]. In the first case $X = \{pt\}$ so $Y = \mathcal{B}$, and ϕ is induced by a unipotent element $u \in G$. We denote Y^ϕ by $\mathcal{B}_u$. It is known [5], [8] that $\dim \mathcal{B}_u = d = \frac{1}{2}(\dim G_u - r)$ where G_u is the centraliser of u and r is the rank of G, and that every component of $\mathcal{B}_u$ has dimension d. Now if G_u° is the connected component of the identity in G_u, the group $C_u = G_u/G_u^\circ$ acts on $H_{2d}(\mathcal{B}_u)$ and the actions of W and C_u commute so we get an action of $W \times C_u$ on $H_{2d}(\mathcal{B}_u)$. Notice also that $H_{2d}(\mathcal{B}_u)$ has a special basis given by the homology classes of the irreducible components of $\mathcal{B}_u$. $G = \mathrm{GL}(N)$, G_u is connected, $H_{2d}(\mathcal{B}_u)$ is an irreducible W-module and any irreducible W-module can be obtained in this fashion. In this way we construct a special basis for any irreducible representation of a symmetric group. We call it the Springer basis.

In the second case X is the closure $\bar{\Omega}$ of Ω being a unipotent class Ω in G, $Y = \bar{\Omega} \times \mathcal{B}$, $\phi : Y \to Y$ is defined by $\phi((u, \bar{B})) = (u, \bar{B}^u)$, $\forall (u, \bar{B}) \in Y$. Then $\dim Y^\phi = \dim G - \frac{1}{2}(\dim G_u + r) = d'$, $u \in \Omega$, and again every irreducible component of Y^ϕ has dimension d'. This case is completely analogous to the first case since it is not hard to see that $H_{2d'}(Y^\phi) \simeq H_{2d}(\mathcal{B}_u)$, $u \in \Omega$, as $W \times C_u$-modules.

In the third case $X = \mathcal{V}$ the variety of unipotent elements in G, $Y = \Gamma = \mathcal{V} \times \mathcal{B} \times \mathcal{B}$, $\phi : \Gamma \to \Gamma$ is defined as $\phi(u, B_1, B_2) = (u, B_1^u, B_2^u)$ $\forall (u, B_1, B_2) \in \Gamma$. In this case $\dim \Gamma^\phi = 2 \dim \mathcal{B} = d$, and again each component of Γ^ϕ had dimension d.

Since the fiber of π is $\mathcal{B} \times \mathcal{B}$ we get an action of $W \times W_0$, W_0 being the opposite Coxeter group to W, and $H_{2d}(\Gamma^\phi)$ is isomorphic as a $W \times W_0$ module to the regular representation $\mathbf{Q}[W]$ under the $W \times W_0$-action induced by left and right multiplication.

In particular, the homology classes of the irreducible components of Γ^ϕ give a special basis for $\mathbf{Q}[W]$. Further, one can give an explicit one-one correspondence between components of Γ^ϕ and elements in W, such that if $w \in W$ and X_w is the component of Γ^ϕ corresponding to w, s is a simple reflection in W (resp. W_0). X_w is vertical with respect to s if $sw < w$ (resp. $ws < w$), in the Bruhart ordering, horizontal otherwise.

Finally let $P, B \subset P \subset G$, be a parabolic subgroup of G. Let W_P be its Weyl group. We want to give a basis for $\mathbf{Q}[W_P \setminus W]$, $\mathbf{Q}[W_P \setminus W]$ being the space of $\mathbf{Q}$-valued functions on $W_P \setminus W$. Let us consider the natural map $\Psi : W \to W_p \setminus W$ which associates to each $w \in W$ its coset $W_P w$. Ψ induces an homomorphism

$$\Psi_* : \mathbf{Q}[W] \to \mathbf{Q}[W_P \setminus W]$$

defined by putting $\Psi_*(\delta_w) = \delta_{\Psi(w)}$ (δ being the delta function). Identify $\mathbf{Q}[W]$ with $H_{2d}(\Gamma^\phi)$.

LEMMA 1.2. *The kernel of* Ψ_* *is spanned by the classes* $\{[X]\}$, X *an irreducible component of* Γ^ϕ, *such that there exists a simple reflection* $s \in W_P$ *for which* $s[X] = -[X]$.

PROOF. Let S_P denote the set of simple reflections in W_P. It is well known that we can identify $W_P \setminus W$ with the elements $w \in W$ such that $sw > w$ for any simple reflection $s \in S_P$. Thus, since the classes $\{[X]\}$ for which there exists an $s \in S_P$ with $s[X] = -[X]$ are in one-one correspondence with the set of $w \in W$ such that there exists an $s \in S_P$ with $sw < w$, it follows that if U denotes the span of those classes, $\dim U = \dim \operatorname{Ker} \Psi_*$. Thus, it is sufficient to prove $U \subseteq \operatorname{Ker} \Psi_*$. But let $[X] \in U$. We have

$$\Psi_*([X]) = \Psi_*(s[X]) = \Psi_*(-[X]) = -\Psi_*([X])$$

so $\Psi_*([X]) = 0$.

Now let us consider $\Gamma_P = \mathcal{V} \times \mathcal{P} \times \mathcal{B}$, $\mathcal{P}$ being the variety of parabolic subgroups conjugate to P, and let ϕ again denote its unipotent transformation $(v, P, B) \to (v, P^v, B^v)$. Consider the canonical projection

$$\pi_P : \Gamma \to \Gamma_P.$$

It is clear that $\pi(\Gamma^\phi) = \Gamma_P^\phi$, so that we get a map $\pi_* : H_*(\Gamma^\phi) \to H_*(\Gamma_P^\phi)$. In particular, since $H_{2d}(\Gamma_P^\phi)$ has a basis given by the classes of its components of maximal dimension and these images of components of Γ^ϕ we have that $\pi_* : H_{2d}(\Gamma^\phi) \to H_{2d}(\Gamma_P^\phi)$ is onto.

COROLLARY 1.3. *$H_{2d}(\Gamma^\phi) \cap \operatorname{Ker} \pi_* = \operatorname{Ker} \Psi_*$. In particular $H_{2d}(\Gamma_P^\phi) \simeq \mathbf{Q}[W_P \setminus W]$ as a right W module.*

PROOF. X is a component of Γ^ϕ such that $[X] \in \operatorname{Ker} \Psi_*$ if and only if there exists $s \in S_P$ such that X is vertical with respect to s. This implies $\dim \pi_P(X) < \dim X$ so $[X] \in \operatorname{Ker} \Psi_*$. Vice versa, it is clear that the components of Γ_P^ϕ of maximal dimension are in one-one correspondence with the components in Γ^ϕ which are horizontal with respect to each $s \in S_P$.

Notice that Lemma 1.2 and its Corollary give a special basis for $\mathbf{Q}[W_P \setminus W]$ given by the homology classes of the components of Γ_P^ϕ of dimension d or equivalently by the images of the homology classes of the components in Γ^ϕ which are horizontal with respect to each $s \in S_P$.

§2. The compatibility of the Springer basis

From now on we restrict ourselves to the case when G is a product of matrix groups. We fix a Borel subgroup B in G for any unipotent conjugacy class $\Omega \in \mathcal{V}/G$ in G, we denote by τ_Ω the set of irreducible components of $\bar{\Omega} \cap B$ and put $\tau^G = \bigcup_\Omega \tau_\Omega$.

We denote by $\rho : \tau^G \to \mathcal{V}/G$ the natural projection.

For any parabolic subgroup P of $G, P \supset B$ we denote by L a Levi subgroup of P and by $\kappa : P \to L$ the natural projection. Let $B^2 =^{\mathrm{def}} \kappa(P)$. Then $B^2 = B \cap L$ and it is a Borel subgroup of L.

For any $X \in \tau^G$ we consider $Y_0 =^{\mathrm{def}} \kappa(X) \subset B^2$. It is clear that for any $y \in Y_0$ and $l \in L$ such that $y^l \in B^2$ we have $y^l \in Y_0$. Since Y_0 is irreducible this implies the existence of $Y \in \tau^L$ such that Y is an open dense subset of Y_0. This construction gives us a map $\chi : \tau^G \to \tau^L$.

For any $\Omega \in \mathcal{V}/G$ we denote by Δ_Ω the product $\bar{\Omega} \times \mathcal{B}$, by ϕ the canonical unipotent transformation on Δ_Ω, and define $\Delta = \bigcup_{\Omega \in \mathcal{V}/G} \Delta_\Omega$. We will identify elements τ^G with irreducible components of Δ^ϕ (see [3]).

Let us now consider $\Gamma = \mathcal{V} \times \mathcal{B} \times \mathcal{B}$ with its obvious unipotent transformation, which we denote again by ϕ. To each component X of the fixpoint set Γ^ϕ we associate a unipotent class $\rho(x) = \Omega \in \mathcal{V}/G$ by letting Ω be the unique unipotent class such that $\Omega \cap p(X)$ is dense in $p(X)$, $p : \Gamma \to \mathcal{V}$ being the projection on the first factor. Let τ_2^G be the set of components of Γ^ϕ. Then we have a natural map $\tau : \tau_2^G \to \tau^G$ induced by the projection $h : \Gamma \to \Delta$ on the first two factors. By composing τ with χ we get a map

$$\chi_2 : \tau_2^G \to \tau^L.$$

From now on we shall concentrate on Γ and χ_2 although all our considerations will hold verbatim for Δ and χ.

Let us give another interpretation of the map χ_2. Let $\mathcal{P}$ be the variety of parabolic subgroups conjugate to P and let

$$\pi : \mathcal{V} \times \mathcal{B} \times \mathcal{B} \to \mathcal{V} \times \mathcal{P} \times \mathcal{B}$$

be the canonical projection. Then π is a fiber bundle whose fibers are isomorphic to $P/B = \mathcal{B}^L$. Let $X \in \tau_2^G$, $\tilde{X} = \pi(X)$. Let $(v, \tilde{P}, \tilde{B}) \in \tilde{X}$ be a generic point of $\tilde{X}$, $\tilde{Y} = \pi^{-1}(v, \tilde{P}, \tilde{B}) \cap X \subset \pi^{-1}((v, \tilde{P}, \tilde{B}))$. Let $g \in G$ be such that $\tilde{P}^g = P$, then conjugation by g induces an isomorphism $\Psi : \pi^{-1}((v, \tilde{P}, \tilde{B})) \to \mathcal{B}^L$. But then $\Psi(\tilde{Y}) = Y_\kappa$ will be an irreducible component of $\mathcal{B}_u^L$, $u = \kappa(v^g)$. We put $\chi_2(X) = Y$.

We leave to the reader the easy verifications of the independence of the above construction from the choice of g and of the fact that this construction is equivalent to that given before.

Let now α be a simple root for G relative to B, P_α the parabolic subgroup generated by B and $U_{-\alpha}$, $s_\alpha \in W$ the simple reflection corresponding to α. Let us recall from Section 1 that we have an action of W on $H_*(\Gamma^\phi)$ such that if $[X] \in H_{2d}(\Gamma^\phi)$, $d = 2 \dim \mathcal{B}$, is the homology class of an irreducible component X of Γ^ϕ, either

$$s_\alpha[X] = -[X] \qquad \text{or}$$

$$s_\alpha[X] = [X] + \sum_{X'} n(X, X', \alpha)[X'],$$

where X' runs over the set $\tau_2^G - \{X\}$. Further, $H_{2d}(\Gamma^\phi)$ with this action is isomorphic to $\mathbf{Q}[W]$ with its natural left action.

LEMMA 2.1. *Suppose X_1, X_2 are two components of Γ^ϕ. Let α be a simple root with $P_\alpha \subset P$ and $n(X_1, X_2, \alpha) \neq 0$. Then if $\chi_2(X_1) = Y_1$, $\chi_2(X_2) = Y_2$ and $\mathcal{O}_1$* (resp.

$\mathcal{O}_2$) is the unique unipotent class in L corresponding to Y_1 (resp. Y_2), *we have*
(a) $\bar{\mathcal{O}}_1 \supseteq \mathcal{O}_2$,
(b) $\mathcal{O}_1 = \mathcal{O}_2$ *if and only if* $\pi(X_1) = \pi(X_2)$,
$\pi : \mathcal{V} \times \mathcal{B} \times \mathcal{B} \to \mathcal{V} \times \mathcal{P} \times \mathcal{B}$ *being the canonical projection.*

PROOF. (a) Since $n(X_1, X_2, \alpha) \neq 0$ we have from Section 1 that X_1 is horizontal with respect to α, X_2 is vertical with respect to α and if $K = X_1 \cap X_2$, $\pi_\alpha(K) = \pi_\alpha(X_2)$, $\pi_\alpha : \mathcal{V} \times \mathcal{B} \times \mathcal{B} \to \mathcal{V} \times \mathcal{P}_\alpha \times \mathcal{B}$ being the canonical projection. In particular, since $P_\alpha \subset P$, $\pi(K) = \pi(X_2)$ and $\pi(K)$ is irreducible. Let $(v, \tilde{P}, \tilde{B}) \in \pi(K)$ be a generic point in $\pi(K)$. As above let g be such that $\tilde{P}^g = P$, then if $u = \phi(v^g)$, $\phi : P \to L$ being the canonical projection, the unipotent class $\mathcal{O}_u$, of u in L, will not depend on the choice of g. Also, since $\pi(K) \subseteq \pi(X_1)$ we have $\bar{\mathcal{O}}_1 \supset \mathcal{O}_u$. Since $\pi(K) = \pi(X_2)$ it follows from our previous construction that $\mathcal{O}_u = \mathcal{O}_2$. This proves (a).

(b) If $\pi(X_2) = \pi(X_1)$ then it is clear that $\mathcal{O}_1 = \mathcal{O}_2$.

Suppose $\mathcal{O}_1 = \mathcal{O}_2 = \mathcal{O}$. Since $\pi(K) = \pi(X_2)$ if $(v, \tilde{P}, \tilde{B}) \in \pi(K)$ is a generic point of $\pi(K)$, $X_2 \cap \pi^{-1}(\tilde{v}, \tilde{P}, \tilde{B})$ by our construction will be isomorphic to a component Y_2 of $\mathcal{B}^L_u$, $u \in \mathcal{O}$. Similarly, if $(\bar{v}, \bar{P}, \bar{B}) \in \pi(X_1)$ is a generic point of $\pi(X_1)$ then $X_1 \cap \pi^{-1}(\tilde{v}, \bar{P}, \bar{B})$ will be isomorphic to a component Y_1 of $\mathcal{B}_u$. But by [4] we know $\dim Y_1 = \dim Y_2$, $\dim X_1 = \dim X_2$ and $\dim \pi(X_2) = \dim X_2 - \dim Y_2 = \dim X_1 - \dim Y_1 = \dim \pi(X_1)$. Since $\pi(X_2) \subset \pi(X_1)$ and $\pi(X_1)$ is irreducible we conclude $\pi(X_1) = \pi(X_2)$.

LEMMA 2.2. *Suppose X_1, X_2 are two components of Γ^ϕ, such that $\pi(X_1) = \pi(X_2)$, $\pi : \mathcal{V} \times \mathcal{B} \times \mathcal{B} \to \mathcal{V} \times \mathcal{P} \times \mathcal{B}$ being the canonical projection. Let $\chi_2(X_1) = Y_1$, $\chi_2(X_2) = Y_2$. Then $n(X_1, X_2, \alpha) = n(Y_1, Y_2, \alpha)$ for any simple root α such that $P_\alpha \subset P$.*

PROOF. We can reduce to the case when X_1 is horizontal with respect to α, since if X_1 is vertical it follows from our second construction of $\chi_2(X_1)$ that also Y_1 will be vertical with respect to α. So

$$s_\alpha[X_1] = -[X_1] \quad \text{and} \quad s_\alpha[Y_1] = -[Y_1].$$

Assume X_1 is horizontal with respect to α, then consider the natural projections

$$\pi_\alpha : \mathcal{V} \times \mathcal{B} \times \mathcal{B} \to \mathcal{V} \times \mathcal{P}_\alpha \times \mathcal{B}, \qquad \pi'_\alpha : \mathcal{V} \times \mathcal{P}_\alpha \times \mathcal{B} \to \mathcal{V} \times \mathcal{P} \times \mathcal{B}.$$

$\pi = \pi'_\alpha \circ \pi_\alpha$, π_α is a $\mathbf{P}^1$-bundle and the generic fibers of $\pi'_\alpha \,|\, x_\alpha(X_1)$ are irreducible since the generic fibers of $r \,|\, X_1$ are. So we can apply Lemma 1.1 with $Z = \pi(X_1) = \pi(X_2)$, $X = X_1$, $Y = \pi_\alpha^{-1}(X)$. So if $\tilde{P} \in Z$ is a generic point of Z

and we put $\tilde{Y}_1 = \pi^{-1}(\tilde{P}) \cap X_1$, $\tilde{Y}_2 = \pi^{-1}(\tilde{P}) \cap X_2$ the lemma implies $n(X_1, X_2, \alpha) = n(\tilde{Y}_1, \tilde{Y}_2, \alpha)$. But by our second construction $n(\tilde{Y}_1, \tilde{Y}_2, \alpha) = n(Y_1, Y_2, \alpha)$, so the lemma follows.

Given our parabolic subgroup $P \supset B$ we are going to define a preorder on τ_2^G as follows. Let $X_1, X_2 \in \tau_2^G$. We set $X_2 <_{\tilde{P}} X_1$ if $\pi(X_2) \subseteq \pi(X_1)$. Notice that if we let $Y_1 = \chi_2(X_1)$, $Y_2 = \chi_2(X_2)$ and $\mathcal{O}_1$ (resp. $\mathcal{O}_2$) be the unipotent class in L corresponding to Y_1 (resp. Y_2), then $\mathcal{O}_2 \subseteq \mathcal{O}_1$.

We say $X_1 \sim_P X_2$ if $X_1 \leqq_P X_2$, $X_2 \leqq_P X_1$. This defines an equivalence relation on τ_2^G and if we denote by $F_P = \tau_2^G/\sim$, the above preorder defines a structure of partially ordered sets on F_P. Further, since if $X_1 \sim_P X_2$ the components $Y_1 = \chi_2(X_1)$, $Y_2 = \chi_2(X_2)$ have the same associated class $\mathcal{O}$ of unipotent elements in L we get a well defined map $\tau : F_p \to \mathcal{V}^L/L$.

THEOREM 2.3. *Let* $R = H_{2d}(\Gamma^\phi) = Q[W]$, $d = 2\dim \mathcal{B}$. *For each* $f_P \in F_P$ *define*

$$\mathcal{L}_{f_P} = \{\text{span in } R \text{ of the classes } [X] \mid X \in f'_P, f'_P \leqq f_P\}.$$

Then

(i) *The family of subspaces* $\{\mathcal{L}_{f_P}\}_{f_P \in F_P}$ *defines a* W_P*-invariant filtration of* R, *the action of* W_P *being the restriction of the left action of* W.

(ii) *For each* $f_P \in F_P$ *consider the space* $M_{f_P} = \mathcal{L}_{f_P}/\Sigma_{f'_P < f_P}\mathcal{L}_{f'_P}$ *together with its basis given by the elements* $\{\mu[X]\}_{X \in f_P}$, $\mu : \mathcal{L}_{f_P} \to M_{f_P}$ *being the quotient homomorphism. Then there exists a* W_P*-equivariant isomorphism*

$$h : M_{f_P} \to H_{2d(\theta)}((\bar{\mathcal{O}} \times \mathcal{B}_L)^\phi)$$

where $\mathcal{O} = \tau(f_P)$ *and* $d(\mathcal{O}) = \dim(\bar{\mathcal{O}} \times \mathcal{B}^L)^\phi$, *such that* $h(\mu[X]) = [Y]$ *where* $Y = \chi_2(X)$ *for each* $X \in f_P$. *So* $M_{f_P} \cong M_{f'_P}$ *if and only if* $\tau(f_P) = \tau(f'_P)$. *In particular,* M_{f_P} *is an absolutely irreducible representation of* W_P.

PROOF. (i) Let $X \in f_P$, and $s_\alpha \in W_P$. We have by Lemma 2.1 that either $s_\alpha[X] = -[X]$ or $s_\alpha[X] = [X] + \Sigma_{X' \leqq_P X} n(X, X', \alpha)[X']$. This proves (i).

(ii) The fact that h is W_P-equivariant is an immediate consequence of Lemma 2.2.

Let us now prove h is an isomorphism. It is sufficient to show that if $X_1 \sim_P X_2$, $\chi_2(X_1) = \chi_2(X_2)$ if and only if $X_1 = X_2$. But since $X_1 \sim_P X_2$ we have $\pi(X_1) = \pi(X_2)$. Further, since $\chi_2(X_1) = \chi_2(X_2)$, $\pi^{-1}(v, \tilde{P}, \tilde{B}) \cap X_1 = \pi^{-1}(v, \tilde{P}, \tilde{B}) \cap X_2$ for a generic point $(v, \tilde{P}, \tilde{B}) \in \pi(X_1)$. This implies $X_1 = X_2$. The last statement is clear.

Finally, $L = \mathrm{GL}(h_1) \times \cdots \times \mathrm{GL}(h_2)$, $h_1 + \cdots + h_2 = N$ and we know that $h(M_{f_P})$ is irreducible, so also is M_{f_P}.

REMARK 2.4. If we consider two parabolic subgroups $Q \supset P \supset B$ then the filtration $\{\mathcal{L}_{f_P}\}_{f_P \in P}$ is a refinement of the filtration $\{\mathcal{L}_{f_Q}\}_{f_Q \in F_Q}$, i.e., each $\mathcal{L}_{f_Q} = \Sigma_{d_{f_P} \subset d_{f_Q}} \mathcal{L}_{f_P}$. In fact, if we consider the natural projections

$$\pi_P : \mathcal{V} \times \mathcal{B} \times \mathcal{B} \rightarrow \mathcal{V} \times \mathcal{B} \times \mathcal{B}, \qquad \pi_Q : \mathcal{V} \times \mathcal{B} \times \mathcal{B} \rightarrow \mathcal{V} \times \mathcal{A} \times \mathcal{B},$$

$$\pi_{P,Q} : \mathcal{V} \times \mathcal{B} \times \mathcal{B} \rightarrow \mathcal{V} \times \mathcal{Q} \times \mathcal{B}$$

then $\pi_Q = \pi_{P,Q} \circ \pi_P$, so given two components X_1, X_2 of $(\mathcal{V} \times \mathcal{B} \times \mathcal{B})^\phi$, $\pi_P(X_2) \subseteq \pi_P(X_1)$ implies $\pi_Q(X_2) \subseteq \pi_Q(X_1)$, hence if $X_2 \leqq_P X_1$, then $X_2 \leqq_Q X_1$. It follows that for each $f_Q \in F_Q$, M_{f_Q} has itself a filtration by W_P-invariant subspaces $\{N_{f_P}\}_{f_P \in F_P}$. Further, $N_{f_P} / \Sigma_{f'_P < f_P} M_{f'_P} = M_{f_P}$, and N_{f_P} has a basis given by the elements of the basis of M_{f_Q} defined above which it contains.

Assume now that $Q = \mathrm{GL}(N)$. Then W_Q is the symmetric group S_N, $W_P = S_{h_1} \times \cdots \times S_{h_2}$, $h_1 + \cdots + h_2 = N$ and U_L/L is in one-one correspondence with the set $\hat{W}_P$ of irreducible representations of W_P. Thus if we order U_L/L by putting $\mathcal{O}_1 \geqq \mathcal{O}_2$ if $\bar{\mathcal{O}}_1 \supseteq \mathcal{O}_2$ we get an ordering on $\hat{W}_P$. Further, the map $\tau : F_P \rightarrow U_L/L$ is compatible with the given ordering on F_P and U_L/L respectively. Thus we can restate Theorem 2.3 and Remark 2.4 as follows.

THEOREM 2.5. *Let U be an irreducible S_N module over* $\mathbf{Q}$. *Then there exists a basis $\{u_1, \cdots, u_s\}$ of U such that:*

(i) *For any partition $\boldsymbol{h} = (h_1, \cdots, h_2)$ of N, if we restrict the action of S_N on U to $S^{\boldsymbol{h}} = S_{h_1} \times \cdots \times S_{h_2}$ and we decompose U as a direct sum $\bigoplus_{\sigma \in \hat{S}^{\boldsymbol{h}}} U_\sigma$, where U_σ is the isotopic component of type σ, then the subspaces $V_\sigma = \bigoplus_{\sigma' \leq \sigma} U_{\sigma'}$ have a basis given by the elements in the basis $\{u_1, \cdots, u_s\}$ which they contain.*

(ii) *Let $W_\sigma = V_\sigma / \Sigma_{\sigma' < \sigma} V_{\sigma'}$, then the image in W_σ of a vector belonging to $(V_\sigma - \Sigma_{\sigma' < \sigma} V_{\sigma'}) \cap \{u_1, \cdots, u_s\}$ generates an irreducible $S^{\boldsymbol{h}}$ module of type σ, which has a basis formed by the images in W_σ of the elements of $(V_\sigma - \Sigma_{\sigma' < \sigma} V_{\sigma'}) \cap \{u_1, \cdots, u_s\}$ which it contains.*

§3. A basis for the irreducible GL(n) modules

Let k be a field of characteristic 0. Let V be a vector space over k, $\dim V = n$, choose a basis $\{e_1, \cdots, e_n\}$ of V so that we can identify $\mathrm{GL}(V)$ with the group of $n \times n$ invertible matrices with entries in k. Let T be the maximal torus of diagonal matrices in $\mathrm{GL}(V)$, B the Borel subgroup of upper triangular matrices.

Let us fix a positive integer N, and consider $V^{\otimes N}$, $\mathrm{GL}(K)$ acts on $V^{\otimes N}$ by $g(v_1 \otimes \cdots \otimes v_N) = gv_1 \otimes \cdots \otimes gv_n$ for any $g \in \mathrm{GL}(V)$. Also S_N acts on $V^{\otimes N}$ by $(v_1 \otimes \cdots \otimes v_N)\sigma = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(N)}$ and it is clear that the two actions commute. In fact, it is a well known fact [9, ch. 4, §4] that $\mathrm{End}_{GL(V)}(V^{\otimes N})$ is spanned by the symmetric group S_N and $\mathrm{End}_{S_N}(V^{\otimes N})$ is spanned by GL(V).

Let $i = (i_1, \cdots, i_n)$, $i_1 + \cdots + i_n = N$, be a partition of N consisting of n possibly empty parts. Given i we define the weight space

$$V_i = \left\{ u \in V^{\otimes N} \,\middle|\, \forall t = \begin{pmatrix} t_1 & 0 \\ 0 & t_n \end{pmatrix} \in T,\ tu = t_1^{i_1} \cdots t_n^{i_n} u \right\}.$$

It is clear that $V^{\oplus N} = \bigoplus_i V_i$, i running over all possible partitions of N consisting of n parts. Also V_i is S_N stable.

We want to study V_i as an S_N-module. V_i has a basis consisting of the vectors $\{e_{t_1} \otimes \cdots \otimes e_{t_n}\}$ with i_1 of the t_n's equal to 1, i_2 of the t_n's equal to 2, $\cdots$, i_n of the t_n's equal to n. It is clear that S_N acts transitively on this basis. Further, if we consider the vector

$$\mathscr{E}_i = \underbrace{e_1 \otimes \cdots \otimes e_1}_{i_1\text{-times}} \cdots \otimes \underbrace{e_n \otimes \cdots \otimes e_n}_{i_n\text{-times}}$$

then S^i is the stabilizer of $\mathscr{E}_i$ in S_N. $S^i = S_{i_1} \times \cdots \times S_{i_n}$, put $S_0 = \mathrm{id}$. This proves that V_i is isomorphic as an S_N-module to $k[S^i \backslash S_N]$.

Now if we consider S_N as the Weyl group of GL(N), and we fix a Borel subgroup $B \subset \mathrm{GL}(N)$. For a suitable parabolic subgroup $P \supset B$. We have $S^i = W_P$. Let $\mathscr{B}$ be the variety of Borel subgroups in GL(N), $\mathscr{P}$ the variety of parabolic subgroups conjugate to P, $\mathscr{V}$ the variety of unipotent elements in GL(V). It follows from our previous results, that if we put $\Gamma_P = \mathscr{V} \times \mathscr{P} \times \mathscr{B}$, and we let ϕ be the obvious unipotent transformation of Γ_P and Γ_P^ϕ its point set, then $k[S^i \backslash S_N] \cong H_{2d}[\Gamma_P^\phi, k)$, $d = 2\dim \mathscr{B}$. Thus, using this isomorphism $k[S^i \backslash S_N]$ has a basis given by the homology classes of the components of Γ_P^ϕ of dimension $2d$. Since V_i is isomorphic to $k[S^i \backslash S_N]$ this gives a basis for V_i and since $V^{\otimes N} = \bigoplus_i V_i$ we get a basis for $V^{\otimes N}$.

We are going to study the properties of this basis. Let $h = (h_1, \cdots, h_r)$, $h_1 + \cdots + h_r = n$, be a partition of n. Write $V = V_{h_1} \oplus \cdots \oplus V_{h_r}$ where V_{h_s} is the subspace spanned by the basic vectors $\{e_{h_1 + \cdots + h_{s-1}+1}, \cdots, e_{h_1 + \cdots + h_s}\}$. Consider the subgroup $G_h = \mathrm{GL}(V_{h_1}) \times \cdots \times \mathrm{GL}(V_{h_2}) \subset \mathrm{GL}(V)$.

Fix a partition $t = (t_1, \cdots, t_r)$, $t_1 + \cdots + t_r = N$, of N consisting of t possibly empty parts. Define W_t as the span of the tensors $v_1 \otimes \cdots \otimes v_N$ with t_1 among the v_h's in V_{h_1}, t_2 among the v_h's in V_{h_2}, $\cdots$, t_r among the v_h's in V_{h_r}. It is clear that $V^{\otimes N} = \bigoplus_t W_t$, t running over all partitions of N consisting of r parts. Further, given $i = (i_1, \cdots, i_n)$, if we let $t(i) = (i_1 + \cdots + i_{h_1}, \cdots, i_{h_1 + \cdots + h_{2-1}+1} + \cdots + i_n)$, we have $V_i \subset W_{t(i)}$ so that, for each $t = (t_1, \cdots, t_r)$, $W_t =$

$\bigoplus_{t(i)=t} V_i$. This implies that W_t has a basis given by the vectors of the basis $V^{\otimes N}$ defined above which it contains.

Now, let us fix $t = (t_1, \cdots, t_r)$ and let us restrict our attention to the i's such that $t(i) = t$.

Consider the subgroups $S^t = S_{t_1} \times \cdots \times S_{t_r} \subset S_N$ and $S^i \subset S_N$.

The fact that $t(i) = t$ implies $S^i \subset S^t$. Further, if we consider S_N as the Weyl group of GL(N) and we fix a Borel subgroup B we can find parabolic subgroups $Q \supset P \supset B$ such that $W_Q = S^t$, $W_P = S^i$. Thus using the construction of the previous section we can find a filtration $\{\mathscr{L}_{f_Q}\}_{f_Q \in F_Q}$ of $k[S_N]$ by S^t stable subspaces each having as basis the vectors in the given basis which it contains. Consider now the quotient map $\psi : S_N \to S^i \backslash S_N$ and let $\psi_* : k[S_N] \to k[S^i \backslash S_N]$ be defined by $\psi_*(\delta_w) = \delta_{\psi(w)}$, δ being the delta function. Put

$$\mathscr{L}^i_{f_Q} = \psi_*(\mathscr{L}_{f_Q}) \qquad \text{for each } f_Q \in F_Q.$$

It is clear that the family of subspaces $\{\mathscr{L}^i_{f_Q}\}_{f_Q \in F_Q}$ defines a filtration of $k[S^i \backslash S_N]$ and that each $\mathscr{L}^i_{f_Q}$ has a basis consisting of the vectors in the given basis of $k[S^i \backslash S^N]$ which it contains. By identifying V_i and $k[S^i \backslash S_N]$ and using the isomorphism given above we can define

$$W_{f_Q} = \bigoplus_{\substack{i \\ t(i)=t}} \mathscr{L}^i_{f_Q} \subset W_t \qquad \text{for each } f_Q \in F_Q.$$

THEOREM 3.1. (i) *The family* $\{W_{f_Q}\}_{f_Q \in F_Q}$ *defines a filtration of* W_t *by* G*-stable subspaces, under the action of* G *which is the restriction of the action of* GL(V) *in* $V^{\otimes N}$.

(ii) *Let* $\mathscr{N}_{f_Q} = W_{f_Q} / \Sigma_{f'_Q < f_Q} W_{f'_Q}$. *Then* $\mathscr{N}_{f_Q}$ *is an irreducible* G*-module or* 0, *and if* L *is the Levi component of* Q, $\tau : F_Q \to U_L / L$ *is the map previously defined at the end of* §2, *then*

$$\mathscr{N}_{f_Q} \cong \mathscr{N}_{f'_Q} \qquad \text{if and only if } \tau(f_Q) = \tau(f'_Q).$$

PROOF. (i) Our statement is an immediate consequence of the definition of W_{f_Q} and Proposition A.1.

We still have to prove (ii) in Theorem 3.1. Recall that an element of V_L / L can be thought of as an r-tuple $((j^{(1)}_1 \geqq \cdots \geqq j^{(1)}_{s_1}), (j^{(2)}_1 \cdots j^{(2)}_{s_2}), \cdots, (j^{(r)}_1 \cdots j^{(r)}_{s_r}))$ of ordered partition of $t_1, \cdots, t_r$ respectively. Since $\Sigma_{m,l} j^{(e)}_m = \Sigma_e t_e = N$, to the above r-tuple of partitions we can associate the partition $(j^{(1)}_1, \cdots, j^{(r)}_{s_r})$ of N.

Thus using the map $\tau : F_Q \to V_L / L$ we can associate to each $f_Q \in F_Q$ a partition j of N which we shall denote by $\tilde{\tau}(f_Q)$.

LEMMA 3.2. *Let* $j = \tilde{\tau}(f_Q)$. *Then*

$$\mathcal{L}^i_{f_Q} \Big/ \sum_{f'_Q < f_Q} \mathcal{L}^i_{f'_Q} \neq 0.$$

PROOF. In order to prove our lemma we have to show that in $\Gamma^\phi = (V \times \mathcal{B} \times \mathcal{B})^\phi$ there is a component $X \in f_Q$ which is horizontal with respect to each simple reflection $s \in S^i$. Let $\tau_2^{\mathrm{GL}(N)}$, τ^L be defined as in the preceding section.

We claim that it is sufficient to show the existence of a component $\tilde{Y} \in \tau^L$ which is horizontal with respect to each simple reflection $s \in S^i$ and whose associated unipotent class is $\tau(f_Q)$. In fact, suppose such a Y exists, then, using the surjectivity of the map $\chi_2 : \tau_2^G \to \tau^L$ we get the existence of a component $Y \in \tau_2^{\mathrm{GL}(N)}$ with $\chi_2(Y) = \tilde{Y}$. Lemma 2.2 implies that Y is horizontal with respect to each simple reflection $s \in S^i$. Further, $Y \in f'_Q$ with $\tau(f'_Q) = \tau(f_Q)$. But then by Theorem 2.3 $M_{f'_Q} = M_{f_Q}$ as S^i-modules. This implies the existence of a component $X \in f_Q$ with the required properties.

In order to find our $\tilde{Y}$ we proceed as follows. By a result of Frobenius [4, 2.24] the multiplicity of M_{f_Q} in $k[S^i \backslash S^t]$ considered as a right S^t module is one. Using our construction we get the existence of a component $Z \subset (\mathcal{V}^L \times \mathcal{B}_i \times \mathcal{B}_i)^{\phi_L}$ which is horizontal with respect to each simple reflection $s \in S^i$ and whose associated unipotent class is $\tau(f_Q)$. Put $\tilde{Y} = \pi(Z)$ where $\pi : \mathcal{V}^L \times \mathcal{B}^L \times \mathcal{B}_2 \to \mathcal{V}^L \times \mathcal{B}^L$ is the projection in the first two factors.

Let us now finish the proof of Theorem 3.1. The above lemma implies that if $\bar{\tau}(f_Q)$ consists of at most n parts then $\mathcal{N}_{f_Q} \neq 0$. Further, if $\mathcal{O} \in \mathcal{V}^L/L$ we have that the set $\tau^{-1}(\mathcal{O})$ will contain $\dim M_{f_Q}(n!/t_1! \cdots t_r!)$ elements, f_Q being any element in $\tau^{-1}(\mathcal{O})$, so if we write, using the linear reductivity of G,

$$W_t = \bigoplus_{f_Q \in F_Q} \mathcal{N}_{f_Q}, \tag{3.1}$$

and if we let $F_Q^{(n)} = \{f_Q \in F_Q \mid \bar{\tau}(f_Q) \text{ consists of at most } n \text{ parts}\}$, we get that the right-hand side of (3.1) will contain at least

$$H = \frac{n!}{t_1! \cdots t_r!} \left(\sum_{f_Q \in F_Q^{(n)}} \dim M_{f_Q} \right)$$

nonzero factors. On the other hand, putting $K = n!/t_1! \cdots t_r!$ we have

$$W_t \cong_G \bigoplus_{i=1}^{K} (V_{n_1}^{\otimes t_1} \otimes \cdots \otimes V_{n_r}^{\otimes t_r})_i$$

and it follows from the classical result in [9, ch. 4, §4] that if we decompose W_t in irreducible G-modules such a decomposition contains exactly H nonzero factors. It follows that each $\mathcal{N}_{f_Q}$ is irreducible or zero, and $\mathcal{N}_{f_Q} \neq 0$ if and only if

$\bar{\tau}(f_Q)$ consists of at most n parts. The last statement of Theorem 3.1 is clear from the above.

REMARK 3.3. It is not hard to see that if $j = \bar{\tau}(f_Q)$ then

$$\dim \mathcal{L}^j_{f_Q} \Big/ \sum_{f'_Q < f_Q} \mathcal{L}^j_{f'_Q} = 1.$$

Also if $w_j \in \mathcal{L}^j_{f_Q} / \Sigma_{f'_Q < f_Q} \mathcal{L}^j_{f'_Q} \subset \mathcal{N}_{f_Q}$, then, if $s > t$, $\alpha_{st} w_j = 0$ for any root vector $\alpha_{st} \in \mathfrak{g}$, $\mathfrak{g}$ being the Lie algebra of G, so w_j is a maximal weight vector with respect to the Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ spanned by the $\alpha_{st} \in \mathfrak{g}$ with $s \geqq t$.

Recall that given an irreducible G-module U then there exists a one-dimensional G-module D such that $U \otimes D$ is isomorphic to an irreducible G-submodule of $V_{n_1}^{\otimes t_1} \otimes V_{n_2}^{\otimes t_2} \otimes \cdots \otimes V_{n_r}^{\otimes t_r}$ for some partition $t = (t_1, \cdots, t_r)$ of a suitable integer $N = \Sigma_{e=1}^r t_e$. Hence $U \otimes D$ is isomorphic to an irreducible G-submodule of $W_t \subset V^{\otimes N}$, hence to one of the $\mathcal{N}_{f_Q}$'s. This allows us to introduce an ordering on the set $\hat{G}$ of isomorphism classes of irreducible representations of G. In fact given $\sigma, \sigma' \in \hat{G}$ let U be an irreducible G module of type σ, U' one of type σ', then we set $\sigma \leqq \sigma'$ if there exists a one-dimensional G-module D such that $U \otimes D \cong \mathcal{N}_{f_Q}$, $U' \otimes D = \mathcal{N}_{f'_Q}$ with $\tau(f_Q) \leqq \tau(f'_Q)$. Using this remark and reasoning exactly like that at the end of Section 2 yields

THEOREM 3.4. *Let U be an irreducible* GL(V) *module. Then there exists a basis $\{u_1, \cdots, u_s\}$ of U such that*:

(i) *For any partition $h = (h_1, \cdots, h_r)$ of n, if we restrict the action of* GL(V) *to $G = \mathrm{GL}(V_{h_1}) \times \cdots \times \mathrm{GL}(V_{h_r})$ and we decompose $U = \bigoplus_{\sigma \in \hat{G}} U$, where U_σ is the isotopic component of type σ, then the subspaces $V_\sigma = \bigoplus_{\sigma' \leqq \sigma} U$ have a basis given by the elements in the basis $\{u_1, \cdots, u_s\}$ which they contain.*

(ii) *Let $W_\sigma = V_\sigma / \Sigma_{\sigma' < \sigma} V_{\sigma'}$, then the image in W_σ of a vector in $(V_\sigma - \Sigma_{\sigma' < \sigma} V_{\sigma'}) \cap \{u_1, \cdots, u_s\}$ generates an irreducible G-module of type σ, which has a basis formed by the images in W_σ of the elements of $(V_\sigma - \Sigma_{\sigma' < \sigma} V_{\sigma'}) \cap \{u_1, \cdots, u_s\}$ which it contains.*

Appendix

PROPOSITION A.1. *Let us keep the notations of Section 3 and let $\mathcal{L} \subset k[S_N]$ be a subspace which is S'-stable under the left action of S'. For each i such that $t(i) = t$ define $\mathcal{L}^i = \psi_x(d) \subset k[S^i S_N]$. Identifying $k[S^i \backslash S_N]$ with V_i put $U = \bigoplus_{t(i)=t} \mathcal{L}^i \subset W_t$.*

Then U is G-stable, under the action of G which is the restriction of the action of GL(V) *on $V^{\otimes N}$.*

PROOF. In order to prove that U is G-stable it is sufficient to show that it is stable under the action of the Lie algebra $\mathfrak{g}$ of G. Since U is clearly stable under the action of the Cartan subalgebra h, it will suffice to show that the root vectors $\{\alpha_{e,e+1}\}$ with $\Sigma_{s=1}^{m-1} h_s + 1 \leqq l \leqq \Sigma_{s=1}^{m} h_s - 1$, and $\{\alpha_{e-1,e}\}$ with $\Sigma_{s=1}^{m-1} h_s + 2 \leqq l \leqq \Sigma_{s=1}^{m} h_s$, for some $1 \leqq m \leqq 2$, leave U stable.

In order to do so we need a few elementary remarks of a general nature. Let M be a finite group $K < H < M$ subgroups. Let $\pi_{H,K} : K \backslash G \rightarrow H \backslash G$ be the quotient map.

We can define

$$\psi_{H,K} : k[K \backslash M] \rightarrow k[H \backslash M]$$

by $\psi_{H,K}(\delta_x) = \delta_{\pi_{H,K}(x)}$ for any $x \in K \backslash M$, δ being the delta function.

And

$$\phi^{K,H} : k[H \backslash M] \rightarrow k[K \backslash M]$$

by $\phi^{K,H}(\delta_x) = \Sigma_{\pi_{K,H}(y)=x} \delta_y$ for any $x \in H \backslash M$.

One gets the following immediate identities whose proof we leave to the reader:

$$\psi_{H,K}\phi^{K,H} = \frac{|H|}{|K|} \cdot \text{id}, \tag{A.2}$$

$$\phi^{\{e\},H}\psi_{H,\{e\}}(T) = \sum_{s \in H} \delta_s \cdot T \qquad \text{for any } T \in k[K \backslash \Gamma], \tag{A.3}$$

being the product in $k[M]$.

Let $S \subset K \subset H$ be another subgroup, then

$$\psi_{H,K} \circ \psi_{K,S} = \psi_{H,S}\,; \qquad \phi^{S,K} \circ \phi^{K,H} = \phi^{S,H}. \tag{A.4}$$

In particular we get

$$\phi^{K,H} = \frac{1}{|K|}\, \psi_{K,\{e\}}\, \phi^{\{e\},H}. \tag{A.5}$$

In fact

$$\frac{1}{|K|}\, \psi_{K,\{e\}}\, \phi^{\{e\},H} = \frac{1}{|K|}\, \psi_{K,\{e\}}\, \phi^{\{e\},K} \phi^{K,H} = \frac{|K|}{|K|}\, \phi^{K,H}.$$

Let us return to the proof of our proposition and note that if $i = (i_1, \cdots, i_h)$

$$\alpha_{p,p+1} V_i = 0 \qquad \text{if } i_p = 0,$$

$$\alpha_{p,p+1} V_i \subset V_j \quad \text{with} \quad j = (i_1, \cdots, i_{p-1}, i_p - 1, i_{p+1} + 1, \cdots, i_n) \quad \text{if} \quad i_p \neq 0.$$

In the first case there is nothing to prove.

Suppose we are in the second case. First note that since $\alpha_{p,p+1} \in \mathfrak{g}$ and W_t is stable under the action of $\mathfrak{g}$, $t = t(i) = t(j)$.

Further, by identifying V_i with $k[S^i \setminus S_N]$, V_j with $k[S^j \setminus S_N]$ one immediately checks that if $k = (i_1, \cdots, i_{p-1}, i_p - 1, 1, i_{p+1}, \cdots, i_n)$, $\alpha_{p,p+1}\big|_{V_i} = \psi_{S^j,S^k}\phi^{S^k,S^i}$. Using identities (A.4) and (A.5) we get

$$\alpha_{p,p+1}\big|_{V_i} = \frac{1}{|S^k|}\,\psi_{S^j,\{e\}}\,\phi^{\{e\},S^i}.$$

It follows from the definition that $\psi_{S^j,\{e\}} = \psi_* : k[S_N] \to k[S^j \setminus S_N]$ maps $\mathscr{L}$ onto $\mathscr{L}^j$. We claim that $\phi^{\{e\},S^i}(\mathscr{L}^i) \subset \mathscr{L}$. Let $T \in \mathscr{L}^i$, $Y \in \mathscr{L}$ be such that $\psi_*(y) = \psi_{S^i,\{e\}}(y) = T$, then by (A.3)

$$\phi^{\{e\},S^i}(x) = \sum_{w \in S^i} \delta_w \cdot y \underset{\text{def}}{=} \sum_{w \in S^i} w[y].$$

Since $S^i \subset S^t$ and $\mathscr{L}$ is stable under the left S^t action this implies $\phi^{\{e\},S^i}(T) \in \mathscr{L}$.

Thus if $\alpha_{p,p+1} \in g$, $\alpha_{p,p+1}(U) \subset U$. In exactly the same fashion one shows that if $\alpha_{p,p-1} \in g$, $\alpha_{p,p-1}(U) \subset U$ and our proposition follows.

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